Week 6

Logarithmatic Differentiation

eg. Find $\frac{d}{dx}(2+\sin x)^{\tan x}$

Note: $\frac{d}{dx} x^a = a x^{a-1}$ ← either base or exponent is a constant

$\frac{d}{dx} a^x = (\ln a) a^x$

The rules cannot be applied directly to this example

Sol Method 1 (Trick: $y = e^{\ln y}$)

$$(2+\sin x)^{\tan x} = e^{\ln(2+\sin x)^{\tan x}}$$

constant

$$= e^{(\tan x)\ln(2+\sin x)}$$

$$\frac{d}{dx}(2+\sin x)^{\tan x} = \frac{d}{dx} e^{(\tan x)\ln(2+\sin x)}$$

$$= e^{(\tan x)\ln(2+\sin x)}\left[(\sec^2 x)\ln(2+\sin x) + \frac{\tan x \cos x}{2+\sin x}\right]$$

$$= (2+\sin x)^{\tan x}\left[(\sec^2 x)\ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right]$$

Same idea but present differently

Method 2 (Logarithmatic differentiation)

Let $y = (2+\sin x)^{\tan x}$

$$\ln y = (\tan x)\ln(2+\sin x)$$

Apply $\frac{d}{dx}$:

$$\frac{1}{y} y' = (\sec^2 x)\ln(2+\sin x) + \frac{\tan x \cos x}{2+\sin x}$$

$$\Rightarrow y' = y\left[(\sec^2 x)\ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right]$$

$$= (2+\sin x)^{\tan x}\left[(\sec^2 x)\ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right]$$

eg. $y = \sqrt{\frac{(x-1)^3(x-6)}{(x+7)^5}}$, $x > 6$

Find y'.

Sol

$$\ln y = \frac{1}{2}\left[3\ln(x-1) + \ln(x-6) - 5\ln(x+7)\right]$$

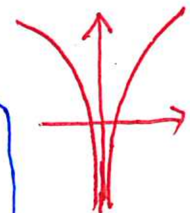
$$\frac{1}{y} y' = \frac{1}{2}\left[\frac{3}{x-1} + \frac{1}{x-6} - \frac{5}{x+7}\right]$$

$$y' = \frac{y}{2}\left[\frac{3}{x-1} + \frac{1}{x-6} - \frac{5}{x+7}\right]$$

Rmk: $x > 6 \Rightarrow$ Every factor > 0

More generally

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \forall x \neq 0}$$



Repeated application of differentiation rules

Chain rule:

eg. $\sin \circ \sqrt{\ } \circ (x^2+x+1)$

$$\frac{d}{dx} \sin\sqrt{x^2+x+1}$$

$$= \underbrace{\left(\cos\sqrt{x^2+x+1}\right) \cdot \frac{1}{2\sqrt{x^2+x+1}} (2x+1)}_{}$$

differentiate layer by layer

Another way: $v = x^2+x+1$

$u = \sqrt{x^2+x+1} = \sqrt{v}$

$y = \sin\sqrt{x^2+x+1} = \sin u$

$$\frac{dy}{dx} = \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dv}} \cdot \boxed{\frac{dv}{dx}}$$

Product rule:

$(fg)' = f'g + fg'$ 1 1

$(fg)'' = f''g + f'g' + f'g' + fg''$

$= f''g + 2f'g' + fg''$ 1 2 1

$(fg)''' = f'''g + f''g' + \dots$

$= f'''g + 3f''g' + 3f'g'' + fg'''$ 1 3 3 1

$$\boxed{\text{Leibniz Rule: } (fg)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} f^{(r)} g^{(n-r)}}$$

Compare this with $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ Binomial theorem

Mean Value Theorems (MVT)

Rolle's theorem

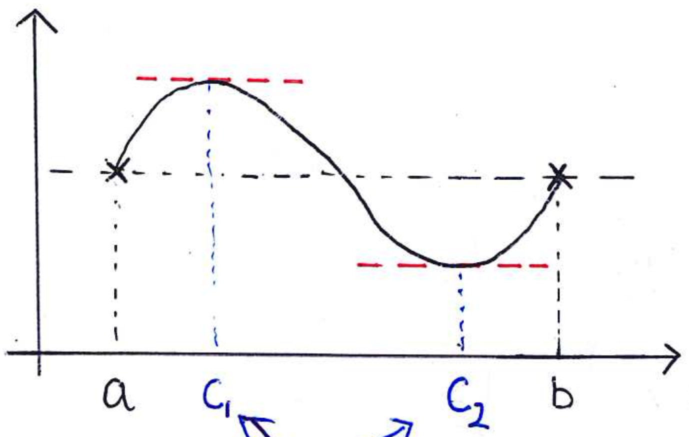
Let f be continuous on $[a,b]$

and differentiable on (a,b)

Also $f(a) = f(b)$

Then $\exists\, c \in (a,b)$ such that

$$f'(c) = 0$$



Two possible choice of c in this example

Pf f is continuous on $[a,b]$

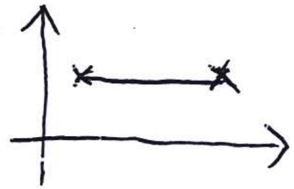
EVT $\Rightarrow$ f has both max and min on $[a,b]$

Case 1 Both max and min occur at the endpoints a, b

Since $f(a) = f(b) \Rightarrow f$ is a constant function!

$\Rightarrow f'(x) \equiv 0$ on $[a,b]$

$\Rightarrow$ We can take any $c \in (a,b)$



Case 2 f has a max / min at a point $c \in (a,b)$

Assume f has a max at c. $\Rightarrow f(x) \le f(c)\ \forall x \in [a,b]$

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$$

$\leftarrow f(c+h) \le f(c) \Rightarrow$ top ≤ 0

$\leftarrow h \to 0^+ \Rightarrow$ bottom > 0

$\Rightarrow \dfrac{\text{top}}{\text{bottom}} \le 0$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$$

$f'(c)$ exists $\Rightarrow f'(c) = 0$

Similarly, $\displaystyle \lim_{h\to 0^-} \frac{\overbrace{f(c+h)-f(c)}^{\le 0}}{\underbrace{h}_{<0}} \ge 0$

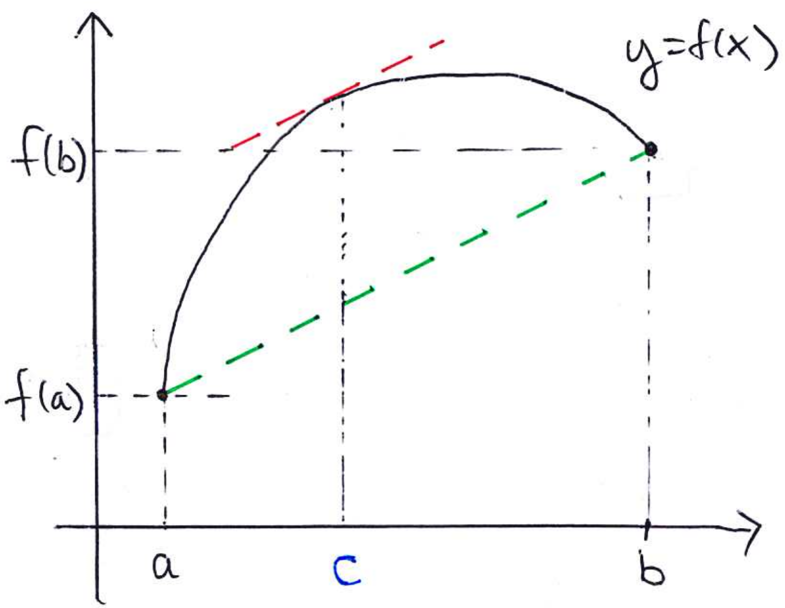
Lagrange's Mean Value Theorem

Let f be continuous on $[a,b]$

and differentiable on (a,b)

Then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Pf Let

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$$

Then

$$g(a) = g(b) = 0, \quad g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

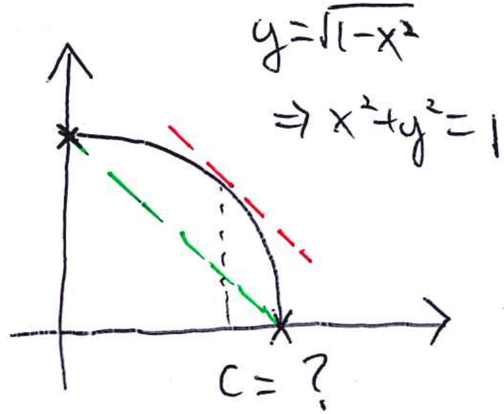
Also, g is continuous on $[a,b]$, differentiable on (a,b)

Rolle's thm $\Rightarrow \exists c \in (a,b)$ such that $g'(c) = 0$

$$\Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

eg Verify Lagrange's MVT for $f(x) = \sqrt{1-x^2}$ on $[0,1]$ by finding such a c.



Sol

$f(x) = \sqrt{1-x^2}$

$f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

$= \frac{-x}{\sqrt{1-x^2}}$

$\frac{f(1)-f(0)}{1-0} = \frac{0-1}{1-0} = -1$

For $f'(x) = \frac{f(1)-f(0)}{1-0} = -1$

$\frac{-x}{\sqrt{1-x^2}} = -1$

$x = \sqrt{1-x^2}$

$\Rightarrow \quad x^2 = 1-x^2$

$x^2 = \frac{1}{2}$

Since we consider

$0 \leq x \leq 1$

take $x = \frac{1}{\sqrt{2}}$

$\therefore \quad c = \frac{1}{\sqrt{2}}$

$f'(\frac{1}{\sqrt{2}}) = \frac{f(1)-f(0)}{1-0}$

eg Show that

$\arctan 3 + \frac{1}{17} < \arctan 4 < \arctan 3 + \frac{1}{10}$

Sol Let $f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2}$

MVT $\Rightarrow \exists\, c \in (3,4)$ such that

$f'(c) = \frac{f(4)-f(3)}{4-3}$

$\frac{1}{17}, \quad \frac{1}{1+c^2} = \frac{\arctan 4 - \arctan 3}{1}$

$3 < c < 4 \Rightarrow \frac{1}{1+4^2} < \frac{1}{1+c^2} < \frac{1}{1+3^2} = \frac{1}{10}$

$\Rightarrow \arctan 3 + \frac{1}{17} < \arctan 4 < \arctan 3 + \frac{1}{10}$

Proposition

Let I be an interval, f is differentiable on I

1. If $f'(x) \equiv 0$ on I, then f is constant on I

2. If $f'(x) \geq 0$ ($f'(x) > 0$) on I, then f is increasing (strictly increasing) on I

3. If $f'(x) \leq 0$ ($f'(x) < 0$) on I, then f is decreasing (strictly decreasing) on I

Rmk Increasing means if $a < b$, then $f(a) \leq f(b)$

Strictly increasing means if $a < b$, then $f(a) < f(b)$

Pf

① Let $a, b \in I$ with $a < b$

MVT $\Rightarrow \exists\, c \in (a,b)$ s.t.

by assumption

$$\frac{f(b) - f(a)}{b - a} = f'(c) \equiv 0$$

$\Rightarrow f(b) = f(a)$

$\Rightarrow f$ is constant on I

② Let $a, b \in I$ with $a < b$.

MVT $\Rightarrow \exists\, c \in (a,b)$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c) \begin{cases} \geq 0 \\ > 0 \end{cases}$$

$$\Rightarrow f(b) - f(a) \begin{cases} \geq 0 \\ > 0 \end{cases}$$

$\Rightarrow f$ is increasing (or strictly increasing) e.g. Show that

$$x^2+\ln(1+x^2) \geq 2x\arctan x \quad \text{for } x \geq 0$$

Sol Let $f(x) = x^2 + \ln(1+x^2) - 2x\arctan x$

$$f'(x) = 2x + \frac{2x}{1+x^2} - 2\arctan x - \frac{2x}{1+x^2}$$

$$= 2x - 2\arctan x$$

$$f''(x) = 2 - \frac{2}{1+x^2} \geq 0 \quad \text{on } [0,\infty)$$

$\Rightarrow$ $f'(x)$ is increasing on $[0,\infty)$

$\Rightarrow$ $f'(x) \geq \underbrace{f'(0)}_{2(0)-2\arctan(0)=0}$ for $x \geq 0$

i.e. $f'(x) \geq 0$ on $[0,\infty)$

$\Rightarrow$ $f(x)$ is increasing on $[0,\infty)$

$\Rightarrow$ $f(x) \geq \underbrace{f(0)}_{0+0+0=0}$ for $x \geq 0$

$\Rightarrow$ $f(x) \geq 0$ for $x \geq 0$

$\Rightarrow$ $x^2+\ln(1+x^2) \geq 2x\arctan x$ for $x \geq 0$

Cauchy's Mean Value Theorem

Let f, g be continuous on $[a, b]$ and differentiable on (a, b)

Also, $g'(x) \neq 0$ on (a, b)

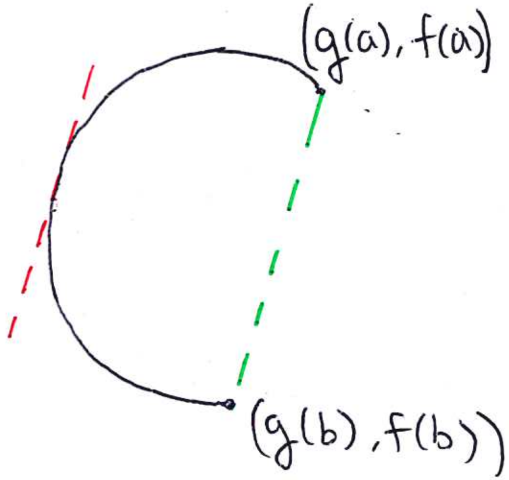
Then $\exists\, c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

e.g.

$f(x) = \cos x$

$g(x) = \sin x$



L'Hôpital Rule

Let $a \in \mathbb{R}$ or $\pm\infty$.

Suppose f, g are differentiable near a

i $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or

$\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$

ii $g'(x) \neq 0$ near a

iii $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or $= \pm\infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Rmk ① Similar version for one-side limit

② Proved using Cauchy's MVT

eg 1

$$\lim_{x\to 1} \frac{x - e^{x-1}}{(x-1)^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{1 - e^{x-1}}{2(x-1)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{-e^{x-1}}{2}$$

$$= \frac{-e^{1-1}}{2}$$

$$= -\frac{1}{2}$$

eg 2

$$\lim_{x\to\infty} \frac{e^{2x}}{x^2+4x+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2e^{2x}}{2x+4} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{4e^{2x}}{2}$$

$$= +\infty$$

Ex $\lim_{x\to\infty} \frac{\ln x}{x^2+3x+7} = 0$

Growth rate to ∞

exponential function > polynomial > logarithm function

eg 3

$$\lim_{x\to\infty} \frac{\sin x + x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\cancel{=} \underbrace{\lim_{x\to\infty} \frac{\cos x + 1}{1}}_{}$$

DNE, not $\pm\infty$

$\Rightarrow$ Cannot apply L'Hôpital's rule

Correct answer For $x>0$,

$$1-\frac{1}{x} < \frac{\sin x + x}{x} = \frac{\sin x}{x} + 1 < 1 + \frac{1}{x}$$

$$\lim_{x\to\infty} 1+\frac{1}{x} = \lim_{x\to\infty} 1-\frac{1}{x} = 1$$

Sandwich theorem

$$\Rightarrow \lim_{x\to\infty} \frac{\sin x + x}{x} = 1$$

Standard form in L'Hopital's Rule: $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

Variations: $\underbrace{0\cdot(\pm\infty) \quad \infty-\infty \quad 1^{\infty} \quad \infty^{0} \quad 0^{0}}$

Strategy: Convert them to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

eg 4

$$\lim_{x\to 0^+} x\ln x \qquad (0\cdot -\infty)$$

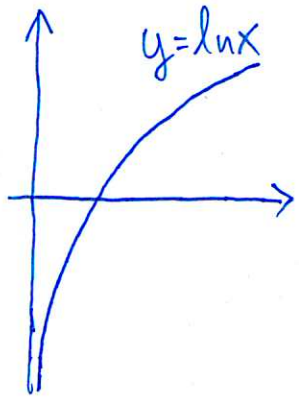
$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \left(\frac{-\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

Easy enough to compare

$$= \lim_{x\to 0^+} -x$$

$$= 0$$



Rmk: If we wrote

$$x\ln x = \frac{x}{\frac{1}{\ln x}}$$

and apply L'Hoptial's things get messy

eg 5

$$\lim_{x\to\frac{\pi}{2}} \sec x - \tan x \qquad (\infty-\infty)$$

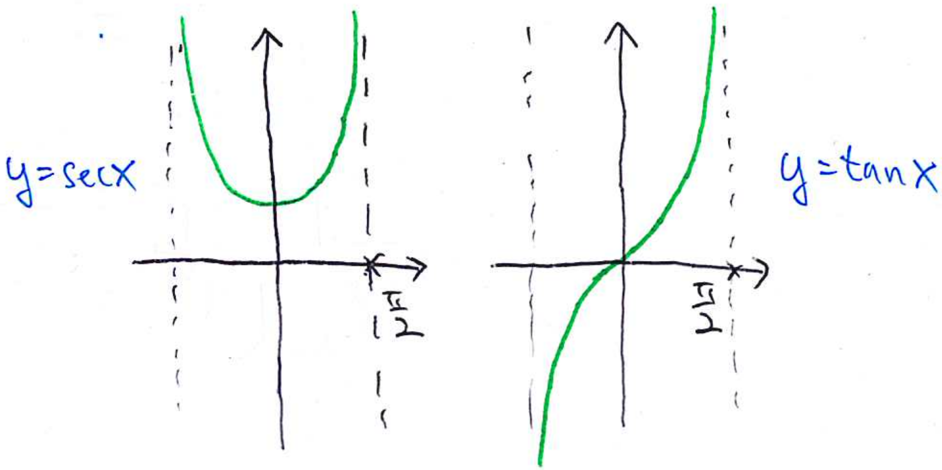
$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \boxed{\frac{-\cos x}{-\sin x}}$$

continuous

$$= \frac{-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = 0$$



eg 6

$$\lim_{x\to 0} (\cos x)^{\csc x} \qquad (1^{\pm\infty})$$

let $y = (\cos x)^{\csc x}$

$$\ln y = (\csc x)(\ln \cos x)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} (\csc x)(\ln \cos x) \quad (\pm\infty \cdot 0)$$

$$= \lim_{x\to 0} \frac{\ln \cos x}{\sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{\cos x}$$

$$= \frac{-\frac{0}{1}}{1} = 0$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0} \ln y} = e^0 = 1$$

exponential function e^z is continuous in z

eg 7

$$\lim_{x\to\infty} x^{\frac{1}{x}} \qquad (\infty^0)$$

let $y = x^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln x$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to 0} \ln y} = e^0 = 1$$

eg8

$$\lim_{x\to 0^+} (1-\cos x)^{\frac{1}{\ln x}} \quad (0^0)$$

Let $y = (1-\cos x)^{\frac{1}{\ln x}}$

$$\ln y = \frac{1}{\ln x} \ln(1-\cos x)$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1-\cos x)}{\ln x} \quad \left(\frac{-\infty}{-\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{x\sin x}{1-\cos x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \frac{\sin x + x\cos x}{\sin x}$$

$$= \lim_{x\to 0^+} \left(1 + \frac{x}{\sin x}\cdot \cos x\right)$$

(or use L'Hôpital once more)

$$= 1 + (1)\cos(0)$$

$$= 2$$

By continuity of exponential function,

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y}$$

$$= e^{\lim_{x\to 0^+} \ln y}$$

$$= e^2$$